MATH 3060 : HW5 Solution

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1. (Generalization of Contraction Mapping Principle)

Let X be a complete metric space. And let $T : X \to X$ be a continuous map such that the k-time composition T^k is contraction. Show that T has a unique fixed point.

Sol) By assumption, $T^k : X \to X$ is a contraction.

By Contraction mapping principle, T^k has a unique fixed point $x \in X$.

Showing x is a fixed point of T: Note that $T^k(Tx) = T(T^k x) = Tx$.

$\therefore$ Tx is also a fixed point of T^k.

By the uniqueness of fixed point of T^k, $Tx = x$.

$\therefore$ x is a fixed point of T.

Showing x is the unique fixed point of T:

Suppose $y \in X$ is also a fixed point of T. Then

$T^k y = T^{k-1}(Ty) = T^{k-1} y = \cdots = Ty = y$. $\therefore$ y is a fixed point of T^k.

By the uniqueness of fixed point of T^k, $y = x$.

Therefore, x is the unique fixed point of T.

2. Show that the equation $\cos x + 2x^4 + x = 1.001$ has a solution near $x=0$.

Sol) Define $\Phi : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ by $\Phi = \mathrm{Id} + \Psi$, where $\Psi(x) = \cos x + 2x^4$ Then $\Phi(0) = 1$.

Applying Perturbation of Identity to Φ: need to construct $r>0$ such that

$\Psi|_{\overline{B_r(0)}} : (\overline{B_r(0)}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is a contraction.

For any $x_1, x_2 \in \overline{B_r(0)}$, $|\Psi(x_2) - \Psi(x_1)| = |(\cos x_2 - \cos x_1) + 2(x_2^4 - x_1^4)|$

$= |(-\sin\xi)(x_2 - x_1) + 2(x_2 - x_1)(x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3)|$ (where ξ is between x_2 and x_1 by applying Mean Value Theorem to $\cos x$.)

$= |(-\sin\xi + 2(x_2^3 + x_2 x_1^2 + x_2^2 x_1 + x_1^3))| \, |x_2 - x_1|$

$\leq (r + 2(r^3 + r^3 + r^3 + r^3)) \, |x_2 - x_1| = (r + 8r^3) \, |x_2 - x_1|$

Choose $r = \frac{1}{4}$; then $\gamma = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} < 1$.

$\therefore$ For all $x_1, x_2 \in \overline{B_{\frac{1}{4}}(0)}$, $|\Psi(x_1) - \Psi(x_2)| \leq \gamma |x_1 - x_2|$. Hence $\Psi|_{\overline{B_{\frac{1}{4}}(0)}}$ is a contraction.

By Perturbation of Identity, for any $y \in \overline{B_R(1)}$, where $R = (1-\gamma)r = \frac{5}{8} \cdot \frac{1}{4} > \frac{1}{10}$,

there exists unique $x \in \overline{B_r(0)}$ such that $\Phi(x) = y$. In particular, $1.001 \in \overline{B_R(1)}$.

$\therefore$ $\cos x + 2x^4 + x = 1.001$ is solvable over $|x| \leq \frac{1}{4}$.

3. Let A be an $n\times n$ symmetric matrix and $v\in\mathbb{R}^n$. Show that there exist $r>0$ and $R>0$ such that $\forall\, y\in\overline{B_R(0)}\subset\mathbb{R}^n$, there exists a unique $x\in\overline{B_r(0)}\subset\mathbb{R}^n$ such that

$$x = y + (x^T A x)v.$$

Sol) Method 1: Apply an alternative of Prop. 3.5.

Prop 3.5' Let $\Phi = Id + \Psi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open subset $U \subseteq \mathbb{R}^n$ containing 0 which satisfies the following properties:

① $\Psi(0) = 0$ ② $\lim_{x\to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0$, $\forall\ 1\leq i,j\leq n$.

Then there exists $r, R>0$ such that for any $y\in \overline{B}_R(0)$, there exists unique $x\in\overline{B}_r(0)$ such that $\Phi(x)=y$.

Pf Almost identical to the proof of Prop. 3.5: in the last step, Apply Thm 3.4 directly instead of its Remark 2. $\square$

Define $\Phi : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^n, \|\cdot\|_2)$ by $\Phi = Id + \Psi$, where $\Psi(x) = -(x^T A x)v$

Then $\Psi(0) = -(0^T A 0)v = 0$. Hence, ① is satisfied.

By Prop. 3.5', it suffices to check ②: $\lim_{x\to 0} \frac{\partial \Phi_i}{\partial x_j}(x) = 0$, for any $1 \le i,j \le n$.

Note that

$$x^T A x = (x_1 \cdots x_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{matrix} x_1 a_{11} x_1 & + \cdots + & x_1 a_{1j} x_j & + \cdots + & x_1 a_{1n} x_n \\ + & \ddots & + & \cdot^{\cdot^{\cdot}} & + \\ x_j a_{j1} x_1 & + \cdots + & x_j a_{jj} x_j & + \cdots + & x_j a_{jn} x_n \\ + & \cdot^{\cdot^{\cdot}} & + & \ddots & + \\ x_n a_{n1} x_1 & + \cdots + & x_n a_{nj} x_j & + \cdots + & x_n a_{nn} x_n \end{matrix}$$

$$\therefore \forall\, 1 \le j \le n,\quad \frac{\partial (x^T A x)}{\partial x_j} = \frac{\partial}{\partial x_j}\Big(x_j a_{jj} x_j + \sum_{\substack{k=1 \\ k\ne j}}^{n} x_k a_{kj} x_j + \sum_{\substack{k=1 \\ k\ne j}}^{n} x_j a_{jk} x_k\Big)$$

$$= 2a_{jj}x_j + \sum_{\substack{k=1 \\ k\ne j}}^{n} a_{kj} x_k + \sum_{\substack{k=1 \\ k\ne j}}^{n} a_{jk} x_k = \sum_{k=1}^{n} (a_{jk} + a_{kj}) x_k .$$

$$\therefore \forall\, 1 \le i,j \le n,\quad \frac{\partial \Phi_i}{\partial x_j}(x) = -\Big(\sum_{k=1}^{n} (a_{jk} + a_{kj}) x_k\Big) v_j . \qquad \therefore \lim_{x\to 0} \frac{\partial \Phi_i}{\partial x_j}(x) = 0$$

Therefore, by Prop. 3.5', there exists $r, R > 0$ such that for any $y \in \overline{B}_R(0)$,

there exists unique $x \in \overline{B}_r(0)$ such that $\Phi(x) = y$, i.e. $x = y + (x^T A x) v$.

Method 2: Apply the perturbation of identity directly.

Applying Perturbation of Identity to Φ defined above: need to construct $r>0$ such that

$\Psi|_{\overline{B}_r(0)} : (\overline{B}_r(0), \|\cdot\|_2) \to (\mathbb{R}^n, \|\cdot\|_2)$ is a contraction.

For any $x, x' \in \overline{B}_r(0)$, $\|\Psi(x) - \Psi(x')\|_2 = \|-(x^T A x) v + ((x')^T A x') v\|_2$

$= \|(-\langle Ax, x\rangle + \langle Ax', x'\rangle) v\|_2$ (where $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the standard inner product.)

$= |-\langle Ax, x\rangle + \langle Ax', x'\rangle| \, \|v\|_2$

$= |\langle A(x'+x), (x'-x)\rangle| \, \|v\|_2$ $\left(\because A \text{ is symmetric} \Rightarrow \langle Ax', -x\rangle = -\langle x', Ax\rangle = -\langle Ax, x'\rangle \right)$

$\leq \|A(x'+x)\|_2 \|x'-x\|_2 \|v\|_2$ (By Cauchy-Schwarz Inequality)

$\leq \|A\|_2 \|x+x'\|_2 \|v\|_2 \|x-x'\|_2$ $\left(\text{where } \|A\|_2 := \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}\right)$

$\leq 2r \|A\|_2 \|v\|_2 \|x-x'\|_2$

Choose $r = \frac{1}{2\|A\|_2\|v\|_2 + 1}$, then $\gamma = \frac{2\|A\|_2\|v\|_2}{2\|A\|_2\|v\|_2+1} < 1$.

$\therefore$ For all $x, x' \in \overline{B}_r(0)$, $\|\Psi(x) - \Psi(x')\|_2 \leq \gamma \|x - x'\|_2$. Hence $\Psi|_{\overline{B}_r(0)}$ is a contraction.

By Perturbation of Identity, for any $y \in \overline{B}_R(0)$, where $R = (1-\gamma) r = \frac{1}{(2\|A\|_2\|v\|_2+1)^2}$.

there exists unique $x \in \overline{B}_r(0)$ such that $\Phi(x) = y$, i.e. $x = y + (x^T A x) \cdot v$.

4. Let $K(x,t) \in C([0,1]\times[0,1])$. Show that there exists $\lambda > 0$ such that for all $g \in C[0,1]$, there exists a unique solution $y \in C[0,1]$ of the integral equation

$$y(x) = g(x) + \lambda \int_0^1 K(x,t)\, y(t)\, dt.$$

Sol) Define $\Phi : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ by $\Phi(y(x)) = y(x) - \Psi(y(x))$,

where $\Psi(y(x)) = -\lambda \int_0^1 K(x,t)\, y(t)\, dt$, where $\lambda > 0$ is to be determined. Then $\Phi(0) = 0$.

Applying Perturbation of Identity to Φ: need to choose $\lambda > 0$ small enough such that $\Psi : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ is a contraction.

For any $y_1, y_2 \in (C[0,1], \|\cdot\|_\infty)$, for any $x \in [0,1]$,

$$|\Psi(y_2(x)) - \Psi(y_1(x))| = \left| -\lambda \int_0^1 K(x,t)\, y_2(t)\, dt + \lambda \int_0^1 K(x,t)\, y_1(t)\, dt \right|$$

$$= \lambda \left| \int_0^1 K(x,t)\, (y_1(t) - y_2(t))\, dt \right| \leq \lambda \cdot \max_{t \in [0,1]} \{|K(x,t)|\} \cdot \|y_2 - y_1\|_\infty.$$

$$\therefore \quad \|\Psi(y_2) - \Psi(y_1)\|_\infty \leq \lambda \cdot \max_{(x,t) \in [0,1]\times[0,1]} \{|K(x,t)|\} \cdot \|y_2 - y_1\|_\infty = \lambda M \|y_2 - y_1\|_\infty.$$

where $M = \max_{(x,t) \in [0,1]\times[0,1]} \{|K(x,t)|\}$. Choose $\lambda = \frac{1}{M+1}$, then $\gamma = \frac{M}{M+1} < 1$.

$\therefore \|\Phi(y_2) - \Phi(y_1)\|_\infty < \gamma \|y_2 - y_1\|_\infty$, for any $y_1, y_2 \in (C[0,1], \|\cdot\|_\infty)$.

$\Phi : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ is a contraction when $\lambda = \frac{1}{M+1}$.

$\therefore$ By Perturbation of Identity, for any $g(x) \in C[0,1]$, there exists $y(x) \in C[0,1]$ such that $\Phi(y(x)) = g(x)$, i.e. $y(x) = g(x) + \lambda \int_0^1 K(x,t) y(t) \, dt$.